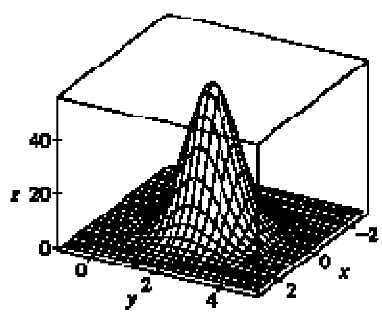
11.7: 4, 8, 10, 46, 48

4. In the figure, points at approximately $(-1,1)$ and $(-1,-1)$ are enclosed by oval-shaped level curves which indicate that as we move away from either point in any direction, the values of f are increasing. Hence we would expect local minima at or near $(-1,\pm1)$. Similarly, the point $(1,0)$ appears to be enclosed by oval-shaped level curves which indicate that as we move away from the point in any direction the values of f are decreasing, so we should have a local maximum there. We also show hyperbola-shaped level curves near the points $(-1,0)$, $(1,1)$, and $(1,-1)$. The values of f increase along some paths leaving these points and decrease in others, so we should have a saddle point at each of these points.

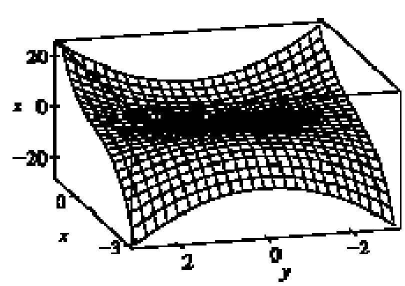
To confirm our predictions, we have $f(x,y) = 3x - x^3 - 2y^2 + y^4 \Rightarrow f_x(x,y) = 3 - 3x^2$, $f_y(x,y) = -4y + 4y^3$. Setting these partial derivatives equal to 0, we have $3 - 3x^2 = 0 \Rightarrow x = \pm1$ and $-4y + 4y^3 = 0 \Rightarrow$ $y(y^2-1) = 0 \Rightarrow y = 0, \pm1$. So our critical points are $(\pm1, 0)$, $(\pm1,\pm1)$. The second partial derivatives are $f_{xx}(x,y) = -6x$, $f_{xy}(x,y) = 0$, and $f_{yy}(x,y) = 12y^2 - 4$, so $D(x,y) = f_{xx}(x,y)\,f_{yy}(x,y) - [f_{xy}(x,y)]^2 = (-6x)(12y^2-4) - (0)^2 = -72xy^2 + 24x$. We use the Second Derivatives Test to classify the 6 critical points:

Critical Point	D	f_{xx}	Conclusion
$(1,0)$	24	-6	$D>0$, $f_{xx}<0 \Rightarrow f$ has a local maximum at $(1,0)$
$(1,1)$	-48		$D<0 \Rightarrow f$ has a saddle point at $(1,1)$
$(1,-1)$	-48		$D<0 \Rightarrow f$ has a saddle point at $(1,-1)$
$(-1,0)$	-24		$D<0 \Rightarrow f$ has a saddle point at $(-1,0)$
$(-1,1)$	48	6	$D>0$, $f_{xx}>0 \Rightarrow f$ has a local minimum at $(-1,1)$
$(-1,-1)$	48	6	$D>0$, $f_{xx}>0 \Rightarrow f$ has a local minimum at $(-1,-1)$

8. $f(x,y) = e^{4y-x^2-y^2} \Rightarrow f_x = -2xe^{4y-x^2-y^2}$, $f_y = (4-2y)e^{4y-x^2-y^2}$, $f_{xx} = (4x^2-2)e^{4y-x^2-y^2}$, $f_{xy} = -2x(4-2y)e^{4y-x^2-y^2}$, $f_{yy} = (4y^2-16y+14)e^{4y-x^2-y^2}$. Then $f_x = 0$ and $f_y = 0$ implies $x = 0$ and $y = 2$, so the only critical point is $(0,2)$. $D(0,2) = (-2e^4)(-2e^4) - 0^2 = 4e^8 > 0$ and $f_{xx}(0,2) = -2e^4 < 0$, so $f(0,2) = e^4$ is a local maximum.

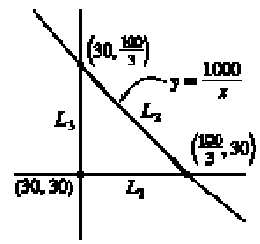


10. $f(x,y) = 2x^3 + xy^2 + 5x^2 + y^2 \Rightarrow f_x = 6x^2 + y^2 + 10x$, $f_y = 2xy + 2y$, $f_{xx} = 12x + 10$, $f_{yy} = 2x + 2$, $f_{xy} = 2y$. Then $f_y = 0$ implies $y = 0$ or $x = -1$. Substituting into $f_x = 0$ gives the critical points $(0,0)$, $(-\frac{5}{3},0)$, $(-1,\pm2)$. Now $D(0,0) = 20 > 0$ and $f_{xx}(0,0) = 10 > 0$, so $f(0,0) = 0$ is a local minimum. Also $f_{xx}(-\frac{5}{3},0) < 0$, $D(-\frac{5}{3},0) > 0$, and $D(-1,\pm2) < 0$. Hence $f(-\frac{5}{3},0) = \frac{125}{27}$ is a local maximum while $(-1,\pm2)$ are saddle points.



46. Let x be the length of the north and south walls, y the length of the east and west walls, and z the height of the building. The heat loss is given by $h = 10(2yz) + 8(2xz) + 1(xy) + 5(xy) = 6xy + 16xz + 20yz$. The volume is 4000 m^3, so $xyz = 4000$, and we substitute $z = \frac{4000}{xy}$ to obtain the heat loss function $h(x, y) = 6xy + 80{,}000/x + 64{,}000/y$.

(a) Since $z = \frac{4000}{xy} \geq 4$, $xy \leq 1000 \quad \Rightarrow \quad y \leq 1000/x$. Also $x \geq 30$ and $y \geq 30$, so the domain of h is $D = \{(x, y) \mid x \geq 30, 30 \leq y \leq 1000/x\}$.



(b) $h(x, y) = 6xy + 80{,}000x^{-1} + 64{,}000y^{-1} \quad \Rightarrow \quad h_x = 6y - 80{,}000x^{-2}$, $h_y = 6x - 64{,}000y^{-2}$.

$h_x = 0$ implies $6x^2y = 80{,}000 \quad \Rightarrow \quad y = \dfrac{80{,}000}{6x^2}$ and substituting into $h_y = 0$ gives $6x = 64{,}000\left(\dfrac{6x^2}{80{,}000}\right)^2 \quad \Rightarrow$

$x^3 = \frac{80{,}000^2}{6 \cdot 64{,}000} = \frac{50{,}000}{3}$, so $x = \sqrt[3]{\frac{50{,}000}{3}} = 10\sqrt[3]{\frac{50}{3}} \quad \Rightarrow \quad y = \frac{80}{\sqrt[3]{60}}$, and the only critical point of h is $\left(10\sqrt[3]{\frac{50}{3}}, \frac{80}{\sqrt[3]{60}}\right) \approx (25.54, 20.43)$ which is not in D.

Next we check the boundary of D. On L_1: $y = 30$, $h(x, 30) = 180x + 80{,}000/x + 6400/3$, $30 \leq x \leq \frac{100}{3}$. Since $h'(x, 30) = 180 - 80{,}000/x^2 > 0$ for $30 \leq x \leq \frac{100}{3}$, $h(x, 30)$ is an increasing function with minimum $h(30, 30) = 10{,}200$ and maximum $h\left(\frac{100}{3}, 30\right) \approx 10{,}533$. On L_2: $y = 1000/x$, $h(x, 1000/x) = 6000 + 64x + 80{,}000/x$, $30 \leq x \leq \frac{100}{3}$. Since $h'(x, 1000/x) = 64 - 80{,}000/x^2 < 0$ for $30 \leq x \leq \frac{100}{3}$, $h(x, 1000/x)$ is a decreasing function with minimum $h\left(\frac{100}{3}, 30\right) \approx 10{,}533$ and maximum $h\left(30, \frac{100}{3}\right) \approx 10{,}587$. On L_3: $x = 30$, $h(30, y) = 180y + 64{,}000/y + 8000/3$, $30 \leq y \leq \frac{100}{3}$. $h'(30, y) = 180 - 64{,}000/y^2 > 0$ for $30 \leq y \leq \frac{100}{3}$, so $h(30, y)$ is an increasing function of y with minimum $h(30, 30) = 10{,}200$ and maximum $h\left(30, \frac{100}{3}\right) \approx 10{,}587$. Thus the absolute minimum of h is $h(30, 30) = 10{,}200$, and the dimensions of the building that minimize heat loss are walls 30 m in length and height $\frac{4000}{30^2} = \frac{40}{9} \approx 4.44$ m.

48. Since $p + q + r = 1$ we can substitute $p = 1 - r - q$ into P giving $P = P(q, r) = 2(1 - r - q)q + 2(1 - r - q)r + 2rq = 2q - 2q^2 + 2r - 2r^2 - 2rq$. Since p, q and r represent proportions and $p + q + r = 1$, we know $q \geq 0$, $r \geq 0$, and $q + r \leq 1$. Thus, we want to find the absolute maximum of the continuous function $P(q, r)$ on the closed set D enclosed by the lines $q = 0$, $r = 0$, and $q + r = 1$. To find any critical points, we set the partial derivatives equal to zero: $P_q(q, r) = 2 - 4q - 2r = 0$ and $P_r(q, r) = 2 - 4r - 2q = 0$. The first equation gives $r = 1 - 2q$, and substituting into the second equation we have $2 - 4(1 - 2q) - 2q = 0 \quad \Rightarrow \quad q = \frac{1}{3}$. Then we have one critical point, $\left(\frac{1}{3}, \frac{1}{3}\right)$, where $P\left(\frac{1}{3}, \frac{1}{3}\right) = \frac{2}{3}$. Next we find the maximum values of P on the boundary of D which consists of three line segments. For the segment given by $r = 0$, $0 \leq q \leq 1$, $P(q, r) = P(q, 0) = 2q - 2q^2$, $0 \leq q \leq 1$. This represents a parabola with maximum value $P\left(\frac{1}{2}, 0\right) = \frac{1}{2}$. On the segment $q = 0$, $0 \leq r \leq 1$ we have $P(0, r) = 2r - 2r^2$, $0 \leq r \leq 1$. This represents a parabola with maximum value $P\left(0, \frac{1}{2}\right) = \frac{1}{2}$. Finally, on the segment $q + r = 1$, $0 \leq q \leq 1$, $P(q, r) = P(q, 1 - q) = 2q - 2q^2$, $0 \leq q \leq 1$ which has a maximum value of $P\left(\frac{1}{2}, \frac{1}{2}\right) = \frac{1}{2}$. Comparing these values with the value of P at the critical point, we see that the absolute maximum value of $P(q, r)$ on D is $\frac{2}{3}$.